

Modeling Crowd Dynamics through Hyperbolic – Elliptic Equations

Rinaldo M. Colombo¹, Maria Gokiel² and Massimiliano D. Rosini³

Abstract. Inspired by the works of Hughes [17, 18], we formalize and prove the well posedness of a hyperbolic–elliptic system whose solutions describe the dynamics of a moving crowd. The resulting model is here shown to be well posed and the time of evacuation from a bounded environment is proved to be finite. This model also provides a microscopic description of the individuals' behaviors.

2010 Mathematics Subject Classification. Primary 35M11; Secondary 35L65, 35J60.

Keywords. Crowd dynamics; hyperbolic–elliptic systems.

1. Introduction

We consider the problem of describing how pedestrians exit an environment. From a macroscopic point of view, we identify the crowd through the pedestrians' density, say $\rho = \rho(t, x)$, and assume that the crowd behavior is well described by the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho V(x, \rho)) = 0, \quad (t, x) \in \mathbb{R}^+ \times \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is the environment available to pedestrians, $V = V(x, \rho) \in \mathbb{R}^2$ is the velocity of the individual at x , given the presence of the density ρ . Several choices for the velocity function are available in the literature, see for instance [5, 6, 8, 9, 11, 17, 18, 19, 25] for velocities depending nonlocally on the density, and [20, Section 4.1] for velocities depending locally on the density. Here, we posit the following (local with respect to the density) assumption:

$$V(x, \rho) = v(\rho) w(x)$$

where $v = v(\rho)$ is a smooth non-increasing scalar function, motivated by the common attitude of moving faster when the density is lower. A key role is played by $w = w(x)$: this vector identifies the route followed by the individual at x . It is reasonable to assume that the individual at x follows the shortest path from x towards the nearest exit. This naturally suggests to choose w parallel to $\nabla \varphi$, the potential φ being the solution to the eikonal equation on Ω . Extending the results in [2, 12, 13] obtained in the 1-dimensional space to the 2-dimensional space, we consider the following elliptic regularization of the eikonal equation:

$$\|\nabla \varphi\|^2 - \delta \Delta \varphi = 1, \quad x \in \Omega,$$

where δ is a fixed strictly positive parameter. Clearly, the resulting vector field $\nabla \varphi$ depends only on Ω , namely only on the geometry of the environment available to the pedestrians, i.e., on the positions of the exits, on the possible presence of obstacles, and so on. We assume that the boundary $\partial\Omega$ is partitioned in walls, say Γ_w , exits, say Γ_e , and corners, say Γ_c ; namely $\partial\Omega = \Gamma_w \cup \Gamma_e \cup \Gamma_c$, the set $\Gamma_e, \Gamma_w, \Gamma_c$ being two by two disjoint. Γ_c is a discrete subset of $\partial\Omega$. Also Γ_e and Γ_w are subsets of $\partial\Omega$ and they are open in the topology they inherit from $\partial\Omega$. It is then natural to choose φ as solution to the elliptic equation

$$\begin{cases} \|\nabla \varphi\|^2 - \delta \Delta \varphi = 1 & x \in \Omega \\ \nabla \varphi(\xi) \cdot \nu(\xi) = 0 & \xi \in \Gamma_w \\ \varphi(\xi) = 0 & \xi \in \Gamma_e, \end{cases} \quad (1.1)$$

$\nu(\xi)$ being the outward unit normal to $\partial\Omega$ at ξ . To select the direction $w(x)$ followed by the pedestrian at x we set

$$w = \mathcal{N}(-\nabla \varphi), \quad (1.2)$$

the map \mathcal{N} being a regularized normalization, that is

$$\mathcal{N}(x) = \frac{x}{\sqrt{\vartheta^2 + \|x\|^2}}, \quad (1.3)$$

for a fixed strictly positive parameter ϑ . Finally, the evolution of the crowd density ρ is then found solving the following scalar conservation law:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v(\rho) w(x)) = 0 & (t, x) \in \mathbb{R}^+ \times \Omega \\ \rho(0, x) = \rho_o(x) & x \in \Omega \\ \rho(t, \xi) = 0 & (t, \xi) \in \mathbb{R}^+ \times \partial\Omega, \end{cases} \quad (1.4)$$

where ρ_o is the initial crowd distribution. In other words, for a given domain Ω , from (1.1) we obtain the vector field $\nabla\varphi$, that is used in (1.2) to define w and then from (1.4) we obtain how the pedestrians' density ρ evolves in time starting from the initial density ρ_o .

Remark that the boundary condition $\rho(t, \xi) = 0$ has to be understood in the sense of conservation laws, see [4, 10] and Definition 2.3 below. Indeed, the choice in (1.4) allows a positive outflow from Ω through Γ_e thanks to the definition of w , as proved in **(E.2)** of Proposition 2.2.

We prove below that the model consisting of (1.1)–(1.2)–(1.4) is well posed, i.e., it admits a unique solution which is a continuous function of the initial data. Moreover, we also ensure that the evacuation time is finite.

Remark that the model (1.1)–(1.2)–(1.4) is completely defined by the physical domain Ω , by the function $v = v(\rho)$ and by the initial datum ρ_o , apart from the regularizing parameters δ and ϑ .

The next two sections are devoted to the detailed formulation of the problem, to the statement of the well posedness result and of further qualitative properties of the model (1.1)–(1.2)–(1.4). All technical details are gathered in Section 4.

2. Well Posedness

Throughout, we denote $\mathbb{R}^+ = [0, \infty[$. For $x \in \mathbb{R}^2$ and $r > 0$, $B(x, r)$ stands for the open disk centered at x with radius r . For any measurable subset S of \mathbb{R}^2 , we denote by $|S|$ its 2-dimensional Lebesgue measure. Recall that two (non-empty) subsets A_1, A_2 of \mathbb{R}^2 are *separate* whenever $\overline{A_1} \cap A_2 = \emptyset = A_1 \cap \overline{A_2}$.

A key role is played by the geometry of the domain Ω . Here we collect the conditions necessary in the sequel, see Figure 1.

($\Omega.1$) $\Omega \subset \mathbb{R}^2$ is non-empty, open, bounded and connected.

($\Omega.2$) The boundary $\partial\Omega$ admits the disjoint decomposition $\partial\Omega = \Gamma_w \cup \Gamma_e \cup \Gamma_c$, where Γ_w and Γ_e are separate and are finite union of open 1-dimensional manifolds of class $\mathbf{C}^{3,\gamma}$, for a given $\gamma \in]0, 1[$; Γ_e is non-empty; Γ_c is a discrete finite set and $\overline{\Gamma_w} \cap \overline{\Gamma_e} \subseteq \Gamma_c \subseteq \overline{\Gamma_w}$.

($\Omega.3$) For any $x \in \Gamma_c$, there exists an $\varepsilon > 0$ such that the intersection $B(x, \varepsilon) \cap \Omega$ is exactly a quadrant of the disk $B(x, \varepsilon)$.

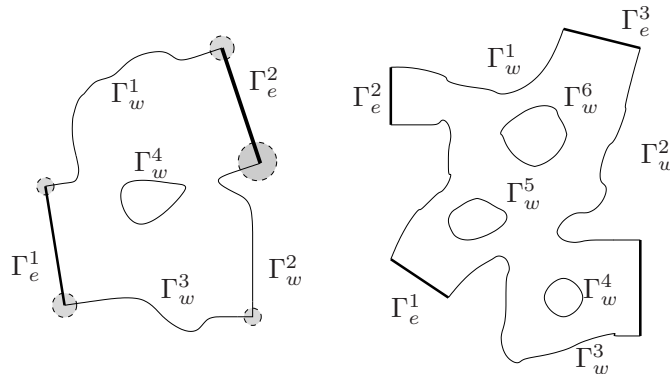


Figure 1. Two examples of sets Ω with the notation used in **($\Omega.2$)** and in **($\Omega.3$)**.

The requirement **($\Omega.1$)** is clear. In **($\Omega.2$)**, the term *open* has to be understood with respect to the topology inherited by $\partial\Omega$. Again concerning **($\Omega.2$)**, introduce the connected components of Γ_w, Γ_e and Γ_c , i.e.,

$$\Gamma_w = \bigcup_{i=1}^{n_w} \Gamma_w^i, \quad \Gamma_e = \bigcup_{i=1}^{n_e} \Gamma_e^i, \quad \text{and} \quad \Gamma_c = \bigcup_{i=1}^{n_c} \{J_i\}.$$

Each of the Γ_e^i is an exit, while the J_i are points where the regularity of $\partial\Omega$ is allowed to be lower. Condition **($\Omega.2$)** implies that each Γ_w^i and each Γ_e^i is a $\mathbf{C}^{3,\gamma}$ manifold. Since $\Gamma_c \subseteq \overline{\Gamma_w}$, along the boundary $\partial\Omega$, between two different exits there is always a wall or, in other words, there can not be two exits separated only by a corner point. Condition **($\Omega.2$)** also implies that $n_e \geq 1$, so that there is at least one exit. Moreover, apart from the trivial case where $\partial\Omega = \Gamma_e$, the set Γ_c may not be empty. Note also that any corner point J_i in Γ_c is either a doorjamb, if $J_i \in \overline{\Gamma_e}$, or a wall corner, if $J_i \in (\overline{\Gamma_w} \setminus \overline{\Gamma_e})$. Condition **($\Omega.3$)** says that the angles between each door and the walls are right and convex, and additionally that these contain straight segments. This is a technical assumption, related to the subtle mixed boundary conditions: Dirichlet and Neumann conditions meet at the doorjamb points. Condition **($\Omega.3$)** ensures the regularity of solutions in a neighborhood of these points, a property that might not hold for general angles.

Throughout, by *solution* to (1.1) we mean *generalized solution* in the sense of the following definition (see [15, Chapters 8 and 13]).

Definition 2.1 *Let Ω satisfy **($\Omega.1$)**. A function $\varphi \in \mathbf{H}^1(\Omega; \mathbb{R})$ is a generalized solution to (1.1) if $\text{tr}_{\Gamma_e} \varphi = 0$ and*

$$\delta \int_{\Omega} \nabla \varphi(x) \cdot \nabla \eta(x) \, dx + \int_{\Omega} \left(\|\nabla \varphi(x)\|^2 - 1 \right) \eta(x) \, dx = 0$$

for any $\eta \in \mathbf{H}^1(\Omega; \mathbb{R})$ such that $\text{tr}_{\Gamma_e} \eta = 0$.

Above, $\text{tr}_{\Gamma_e} \eta$ denotes the trace of η on Γ_e . We refer to [14, Chapter 5.5] for the definition and properties of the trace operator.

Note that no generalized solution to (1.1) can vanish a.e. on Ω . The next proposition provides the basic existence result for the solutions to (1.1), together with some qualitative properties.

Proposition 2.2 (Elliptic Problem) *Let Ω satisfy **($\Omega.1$)**, **($\Omega.2$)**, **($\Omega.3$)**. Fix $\delta > 0$. Then, problem (1.1) admits a unique generalized solution $\varphi \in \mathbf{C}^3(\overline{\Omega}; \mathbb{R})$ with the properties:*

(E.1) *For a.e. $x \in \Omega$, $\nabla \varphi(x) \neq 0$.*

(E.2) *For all $\xi \in \Gamma_e$, $-\nabla \varphi(\xi) \cdot \nu(\xi) > 0$.*

(E.3) $\frac{|\Omega|}{\delta} \exp\left(-\frac{\max_{\partial\Omega} \varphi}{\delta}\right) \leq -\int_{\Gamma_e} \nabla \varphi(\xi) \cdot \nu(\xi) \, d\xi \leq \frac{|\Omega|}{\delta} \exp\left(\frac{\max_{\partial\Omega} \varphi}{\delta}\right).$

The proof of the above proposition is postponed to Section 4. Here, we note that properties **(E.1)**, **(E.2)** and **(E.3)** have clear consequences on the properties of the solutions to the full system (1.1)–(1.2)–(1.4). Indeed, setting w as in (1.2), property **(E.1)** implies that w vanishes only on a set of measure 0; **(E.2)** ensures that w is non zero and points outwards along exits; **(E.3)** can be used to provide bounds on the evacuation time.

In the hyperbolic problem (1.4), we use the following assumptions, which are standard in the framework of conservation laws:

(C.1) $v \in \mathbf{C}^2([0, R_{\max}]; [0, V_{\max}])$ is weakly decreasing, $v(0) = V_{\max}$ and $v(R_{\max}) = 0$.

(C.2) $\rho_o \in (\mathbf{BV} \cap \mathbf{L}^\infty)(\Omega; [0, R_{\max}])$.

Above, R_{\max} , respectively V_{\max} , is the maximal density, respectively speed, possibly reached by the pedestrians.

We recall also the definition of entropy solution to (1.4), which originates in [27], see also [4, p. 1028]. Here, we refer to [10, Definition 2.1].

Definition 2.3 *Let the conditions **($\Omega.1$)**, **($\Omega.2$)**, **(C.1)** and **(C.2)** hold. Let $w \in \mathbf{C}^2(\overline{\Omega}; \overline{B}(0, 1))$. A function $\rho \in (\mathbf{L}^\infty \cap \mathbf{BV})([0, T] \times \Omega; [0, R_{\max}])$ is an entropy solution to the initial – boundary value problem (1.4) if for any test function $\zeta \in \mathbf{C}_c^2([-\infty, T] \times \mathbb{R}^2; \mathbb{R}^+)$ and for any $k \in [0, R_{\max}]$*

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ |\rho(t, x) - k| \partial_t \zeta(t, x) + \text{sign}(\rho(t, x) - k) \left(\rho(t, x) v(\rho(t, x)) - k v(k) \right) w(x) \cdot \nabla \zeta(t, x) \right\} dx \, dt \\ & + \int_{\Omega} |\rho_o(x) - k| \zeta(0, x) \, dx + \int_0^T \int_{\partial\Omega} \left(\text{tr}_{\partial\Omega} \rho(t, \xi) v \left(\text{tr}_{\partial\Omega} \rho(t, \xi) \right) - k v(k) \right) w(\xi) \cdot \nu(\xi) \zeta(t, \xi) \, d\xi \, dt \geq 0. \end{aligned}$$

As above, $\text{tr}_{\partial\Omega} u$ stands for the operator trace at $\partial\Omega$ applied to the \mathbf{BV} function u , see for instance [14, § 5.5] or [10, Appendix]. Note that if the solution has bounded total variation in time, it has a trace at $t = 0+$.

Proposition 2.4 (Hyperbolic Problem) *Let the conditions $(\Omega.1)$, $(\Omega.2)$ and $(C.1)$ hold. Let $w \in \mathbf{C}^2(\bar{\Omega}; \bar{B}(0, 1))$. Then, problem (1.4) generates the map*

$$\begin{array}{ccc} \mathcal{S} & : & \mathbb{R}^+ \times (\mathbf{L}^1 \cap \mathbf{BV})(\Omega; [0, R_{\max}]) \rightarrow (\mathbf{L}^1 \cap \mathbf{BV})(\Omega; [0, R_{\max}]) \\ & & t, \quad \rho \quad \mapsto \quad \mathcal{S}_t \rho \end{array}$$

with the following properties:

(H.1) \mathcal{S} is a semigroup.

(H.2) \mathcal{S} is Lipschitz continuous with respect to the \mathbf{L}^1 -norm, more precisely for any $s, t \in [0, T]$

$$\|\mathcal{S}_t \rho_o - \mathcal{S}_s \rho_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \left[\sup_{\tau \in [s, t]} \text{TV}(\mathcal{S}_\tau \rho_o) \right] |t - s|.$$

(H.3) For any $t \in [0, T]$

$$\|\mathcal{S}_t \rho_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \|\rho_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \exp(C_1 t), \quad \text{TV}(\mathcal{S}_t \rho_o) \leq C_2 (1 + t + \text{TV}(\rho_o)) \exp(C_2 t),$$

where the constants C_1, C_2 depend only on R_{\max} , $\|v'\|_{\mathbf{W}^{2, \infty}([0, R_{\max}]; \mathbb{R})}$ and $\|w\|_{\mathbf{W}^{2, \infty}(\Omega; \mathbb{R}^2)}$.

(H.4) For any $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\Omega; [0, R_{\max}])$, the orbit $t \mapsto \mathcal{S}_t \rho_o$ is the unique solution to (1.4) in the sense of Definition 2.3.

The proof of the above proposition is deferred to Section 4, where it is shown that the above statements follow from [10, Theorem 2.7].

We now give the definition of solution to (1.1)–(1.2)–(1.4).

Definition 2.5 *Let the assumptions $(\Omega.1)$, $(\Omega.2)$, $(\Omega.3)$, $(C.1)$ and $(C.2)$ hold. The pair of functions $(\varphi, \rho) \in \mathbf{H}^1(\Omega; \mathbb{R}) \times (\mathbf{L}^\infty \cap \mathbf{BV})([0, T] \times \Omega; [0, R_{\max}])$ solves the problem (1.1)–(1.2)–(1.4) if φ is a generalized solution to (1.1) in the sense of Definition 2.1 and ρ is an entropy solution to (1.4) in the sense of Definition 2.3 with w given by (1.2).*

The next theorem ensures the well posedness of the elliptic–hyperbolic model (1.1)–(1.2)–(1.4).

Theorem 2.6 (Mixed Problem) *Let the conditions $(\Omega.1)$, $(\Omega.2)$, $(\Omega.3)$, $(C.1)$ and $(C.2)$ hold. For any $\delta, \vartheta > 0$, the elliptic–hyperbolic problem (1.1)–(1.2)–(1.4) generates a map*

$$\begin{array}{ccc} \mathcal{M} & : & \mathbb{R}^+ \times (\mathbf{L}^1 \cap \mathbf{BV})(\Omega; [0, R_{\max}]) \rightarrow (\mathbf{L}^1 \cap \mathbf{BV})(\Omega; [0, R_{\max}]) \\ & & t, \quad \rho \quad \mapsto \quad \mathcal{M}_t \rho \end{array}$$

with the following properties:

(M.1) \mathcal{M} is a semigroup.

(M.2) \mathcal{M} is Lipschitz continuous with respect to the \mathbf{L}^1 -norm, more precisely for any $s, t \in [0, T]$

$$\|\mathcal{M}_t \rho_o - \mathcal{M}_s \rho_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \left[\sup_{\tau \in [s, t]} \text{TV}(\mathcal{M}_\tau \rho_o) \right] |t - s|.$$

(M.3) For any $t \in [0, T]$ we have that $(\varphi, \rho) = \mathcal{M}_t \rho_o$ satisfies

$$\|\rho\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \|\rho_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \exp(C_1 t), \quad \text{TV}(\rho) \leq C_2 (1 + t + \text{TV}(\rho_o)) \exp(C_2 t),$$

where C_1 is a positive constant depending on $\|q\|_{\mathbf{W}^{1, \infty}([0, R_{\max}]; \mathbb{R})}$ and $\|w\|_{\mathbf{W}^{1, \infty}(\Omega; \mathbb{R}^2)}$, while the constant C_2 depends on $\|q\|_{\mathbf{W}^{2, \infty}([0, R_{\max}]; \mathbb{R})}$ and $\|w\|_{\mathbf{W}^{2, \infty}(\Omega; \mathbb{R}^2)}$, where as usual we set $q(\rho) = \rho v(\rho)$.

(M.4) For all $\rho_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\Omega; [0, R_{\max}])$, the orbit $t \mapsto \mathcal{M}_t \rho_o$ is the unique solution to (1.1)–(1.2)–(1.4) in the sense of Definition 2.5.

The above result is a direct consequence of Proposition 2.2 and Proposition 2.4.

3. Qualitative Properties

Here, we aim at further qualitative properties of the solutions to (1.1)–(1.2)–(1.4) that have a relevant meaning in the present setting.

Introduce for $\hat{x} \in \Omega$ the path $p_{\hat{x}}$ followed by those pedestrians that are at \hat{x} at time $t = 0$, i.e., the map $p_{\hat{x}}$ is defined for $t \geq 0$ as the solution to the Cauchy problem

$$\begin{cases} \dot{x} = w(x) \\ x(0) = \hat{x}, \end{cases} \quad \text{where} \quad w = \mathcal{N}(-\nabla\varphi). \quad (3.1)$$

Above, \mathcal{N} is defined in (1.3) and φ is the solution to (1.1).

Proposition 3.1 (Pedestrians' Trajectories) *Let Ω satisfy $(\Omega.1)$, $(\Omega.2)$, $(\Omega.3)$ and call φ the solution to (1.1) provided by Proposition 2.2. Then:*

(Q.1) *For any $\hat{x} \in \Omega$, there exists a unique globally defined path $p_{\hat{x}}: I_{\hat{x}} \rightarrow \mathbb{R}^2$ solving (3.1), $I_{\hat{x}}$ being a suitable non trivial real interval.*

(Q.2) *Any two paths either coincide or do not intersect, in the sense that for any $\hat{x}, \hat{y} \in \Omega$*

$$p_{\hat{x}}(I_{\hat{x}}) \cap p_{\hat{y}}(I_{\hat{y}}) \neq \emptyset \implies \begin{cases} \text{either} & \hat{x} \in p_{\hat{y}}(I_{\hat{y}}) \text{ and } p_{\hat{x}}(I_{\hat{x}}) \subseteq p_{\hat{y}}(I_{\hat{y}}) \\ \text{or} & \hat{y} \in p_{\hat{x}}(I_{\hat{x}}) \text{ and } p_{\hat{y}}(I_{\hat{y}}) \subseteq p_{\hat{x}}(I_{\hat{x}}). \end{cases}$$

(Q.3) *There exist a subset $\hat{\Omega} \subset \Omega$ with $|\hat{\Omega}| = 0$ and a map $T: \Omega \setminus \hat{\Omega} \rightarrow \mathbb{R}^+$ such that $I_{\hat{x}} = [0, T_{\hat{x}}]$ and $p_{\hat{x}}(T_{\hat{x}}) \in \Gamma_e$ for all $x \in \Omega \setminus \hat{\Omega}$.*

The proof is deferred to Section 4. In other words, $T_{\hat{x}}$ is the time that the pedestrian leaving from point \hat{x} needs to reach the exit. Property **(Q.3)** ensures that this time is finite for a.e. initial position \hat{x} . Figure 2

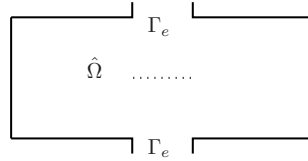


Figure 2. An example in which the set $\hat{\Omega}$ in Proposition 3.1 is necessarily non empty. In the room above, due to the presence of the two exits Γ_e , the vector field w vanishes along the dotted segment $\hat{\Omega}$.

shows that the set $\hat{\Omega}$ may not be avoided under the present assumptions.

4. Technical Details

We choose the following notation to denote a vector orthogonal to a given vector in \mathbb{R}^2 :

$$\text{if } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \text{then } v^\perp = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}.$$

We frequently use the boundedness and Lipschitz continuity of the map \mathcal{N} as defined in (1.3), namely

$$\begin{aligned} \|\mathcal{N}(x)\| &\leq 1 && \text{for all } x \in \mathbb{R}^2, \\ \|\mathcal{N}(x_1) - \mathcal{N}(x_2)\| &\leq \vartheta^{-1} \|x_1 - x_2\| && \text{for all } x_1, x_2 \in \mathbb{R}^2. \end{aligned} \quad (4.1)$$

The Hopf-Cole transformation (see e.g. [14, Chapter 4.4.1])

$$u = e^{-\varphi/\delta} \quad (4.2)$$

transforms generalized solutions to (1.1) into generalized solutions to the *linear* problem

$$\begin{cases} u = \delta^2 \Delta u & x \in \Omega \\ \nabla u(\xi) \cdot \nu(\xi) = 0 & \xi \in \Gamma_w \\ u(\xi) = 1 & \xi \in \Gamma_e, \end{cases} \quad (4.3)$$

whose precise definition (see e.g. [15, Chapter 8]) is here below.

Definition 4.1 A function $u \in \mathbf{H}^1(\Omega; \mathbb{R})$ is a generalized solution to (4.3) on Ω if $\text{tr}_{\Gamma_e} u \equiv 1$ and

$$\delta^2 \int_{\Omega} \nabla u(x) \cdot \nabla \eta(x) \, dx + \int_{\Omega} u(x) \eta(x) \, dx = 0 \quad (4.4)$$

for any $\eta \in \mathbf{H}^1(\Omega; \mathbb{R})$ such that $\text{tr}_{\Gamma_e} \eta \equiv 0$.

The next Lemma collects various information on (4.3).

Lemma 4.2 Fix a positive δ and let Ω satisfy **($\Omega.1$)** and **($\Omega.2$)**. Then,

(u.1) Problem (4.3) admits a unique generalized solution $u \in (\mathbf{H}^1 \cap \mathbf{C}^\infty)(\Omega; \mathbb{R})$ in the sense of Definition 4.1. Moreover, $u \in \mathbf{C}^3(\overline{\Omega} \setminus \Gamma_e; \mathbb{R})$.

(u.2) There exists a positive ϖ dependent only on Ω such that $u(x) \in]\varpi, 1[$ for all $x \in \Omega$, so that $u(x) \in [\varpi, 1]$ also for all $x \in \overline{\Omega}$.

(u.3) The solution u to (4.3) satisfies $\nabla u(\xi) \cdot \nu(\xi) > 0$ for all $\xi \in \Gamma_e$.

(u.4) The set $\{x \in \Omega: \nabla u(x) = 0\}$ of critical points of u has measure 0.

If in addition Ω satisfies **($\Omega.3$)**, then:

(u.5) $u \in \mathbf{C}^3(\overline{\Omega}; \mathbb{R})$.

(u.6) If $\bar{x} \in \overline{\Omega}$ is a critical point of u , then the Hessian matrix $D^2u(\bar{x})$ has at least one positive eigenvalue.

Proof. Consider the different items above separately.

★ **(u.1):** we use Lax–Milgram Lemma, see [14, Section 6.2.1]. Introduce the Hilbert space $H = \{\eta \in \mathbf{H}^1(\Omega; \mathbb{R}): \text{tr}_{\Gamma_e} \eta = 0 \text{ a.e. on } \Gamma_e\}$ endowed with the usual scalar product and the coercive bilinear form

$$a(u, \eta) = \delta^2 \int_{\Omega} \nabla u(x) \cdot \nabla \eta(x) \, dx + \int_{\Omega} u(x) \eta(x) \, dx.$$

Note that H is a closed subspace of $\mathbf{H}^1(\Omega; \mathbb{R})$ by the Trace Theorem [14, Chapter 5.5, Theorem 1]. Indeed, if u^k is a sequence in H converging to u in $\mathbf{H}^1(\Omega; \mathbb{R})$, then

$$\|u\|_{\mathbf{L}^2(\Gamma_e; \mathbb{R})} = \|u^k - u\|_{\mathbf{L}^2(\Gamma_e; \mathbb{R})} \leq C \|u^k - u\|_{\mathbf{H}^1(\Omega; \mathbb{R})} \rightarrow 0,$$

for a constant C depending only on Ω , so that $u \in H$. A function $u \in \mathbf{H}^1(\Omega; \mathbb{R})$ is a generalized solution to (4.3) if and only if $v = u - 1 \in H$ and $a(v, \eta) = -\int_{\Omega} \eta(x) \, dx$ for all $\eta \in H$. The map $\eta \mapsto \int_{\Omega} \eta(x) \, dx$ is a linear functional over H . By Lax–Milgram Lemma, we infer the existence and uniqueness of a generalized solution u to (4.3) such that $u \in H \subset \mathbf{H}^1(\Omega; \mathbb{R})$. Moreover, $u \in \mathbf{C}^\infty(\Omega; \mathbb{R})$ by [14, Theorem 3 in Chapter 6.3 and Theorem 6 in Section 5.6.3]. By **($\Omega.1$)** and **($\Omega.2$)**, the results in [1, Theorem 9.3] ensure that $u \in \mathbf{C}^3(\overline{\Omega} \setminus \Gamma_e; \mathbb{R})$.

★ **(u.2):** note that, due to the boundary conditions along Γ_e and Γ_w , no \mathbf{H}^1 solution to (4.3) can be constant. The function $\eta = (u - 1)^+$, where $(v)^+ = \max(v, 0)$, is in $\mathbf{H}^1(\Omega; \mathbb{R})$ and inserting it in (4.4) we get

$$\delta^2 \int_{\Omega} \|\nabla(u - 1)^+\|^2 + \int_{\Omega} |(u - 1)^+|^2 + \int_{\Omega} (u - 1)^+ = 0.$$

This leads to $(u - 1)^+ \equiv 0$ a.e. in Ω , and, by the continuity of u on $\overline{\Omega}$, $u(x) \leq 1$ for all $x \in \overline{\Omega}$. The map u satisfies (4.3) in the strong sense everywhere in Ω . Hence, by the maximum principle [23, Chapter 2, Theorem 6] $u(x) < 1$ for all $x \in \Omega$.

We show now that $u > 0$. As u is continuous in $\overline{\Omega}$, it attains its minimum. Assume, by contradiction, that $\min_{\overline{\Omega}} u = -m$ for some $m \geq 0$. Then, by applying the maximum principle to $-u$, we know that there exists $\xi \in \partial\Omega$ such that $u(\xi) = -m$. We apply now Hopf’s Lemma, more precisely its extension from [21] to domains satisfying the cone condition (instead of the ball condition as in the original work by Hopf, see e.g. [23, Theorem 8 in Chapter 2]), which implies that the normal derivative of u at ξ is positive, contradicting (4.3).

★ **(u.3):** is an immediate consequence of **(u.2)**, due to the boundary conditions in (4.3).

★ **(u.4):** denote by D^2u the Hessian matrix of u and note that

$$\{x \in \Omega: \nabla u(x) = 0\} = \{x \in \Omega: \nabla u(x) = 0 \text{ and } \det D^2u(x) = 0\} \cup \{x \in \Omega: \nabla u(x) = 0 \text{ and } \det D^2u(x) \neq 0\}.$$

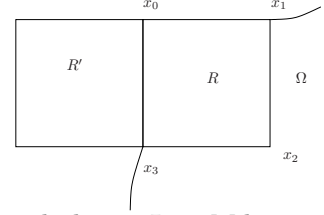
The former set has 2-dimensional measure zero by Sard Theorem [24] applied to ∇u . The latter set consists of isolated points all belonging to the compact set $\bar{\Omega}$, hence it is finite. Therefore, $|\{x \in \Omega : \nabla u(x) = 0\}| = 0$.
★ (u.5): we verify that u is \mathbf{C}^3 at the points in Γ_c under condition **(Ω.3)**. To this aim, we adapt the arguments in [26, Proof of Theorem 3.1], there applied to Poisson equation.

Fix $x_o \in \Gamma_c \cap \bar{\Gamma}_e$, i.e., x_o is a doorjamb. Let ε be as in **(Ω.3)**, call $\ell = \varepsilon/2$ and choose $x_1 \in \Gamma_e \cap B(x_o, \ell)$ with $x_1 \neq x_o$. Let ν be a unit vector such that $\nu \cdot (x_1 - x_o) = 0$ and pointing outward Ω at x_1 . Define $x_2 = x_1 - \ell \nu$ and $x_3 = x_o - \ell \nu$. Call R the open rectangle with vertexes x_o, x_1, x_2, x_3 , denote by $x_i x_j$ the open segment

$$x_i x_j = \left\{ x \in \mathbb{R}^2 : x = (1 - \vartheta) x_i + \vartheta x_j, \vartheta \in]0, 1[\right\}$$

and by \mathfrak{S} the symmetry about the straight line including $x_o x_3$ and $R' = \mathfrak{S}(R)$. Define the rectangle $\mathcal{R} = R \cup x_o x_3 \cup R'$ and consider the problem

$$\begin{cases} -\delta^2 \Delta w(x) + w(x) = 0 & x \in \mathcal{R} \\ w(\xi) = 1 & \xi \in x_o x_1 \cup \mathfrak{S}(x_o x_1) \\ w(\xi) = u(\xi) & \xi \in x_1 x_2 \cup x_2 x_3 \\ w(\xi) = w(\mathfrak{S}(\xi)) & \xi \in \mathfrak{S}(x_1 x_2 \cup x_2 x_3) . \end{cases}$$



Note that the boundary condition is of class \mathbf{C}^∞ by the regularity of u proved above. Lax–Milgram Lemma ensures that the function w exists, is unique and is in $\mathbf{C}^\infty(\mathcal{R}; \mathbb{R})$. By construction, w is symmetric with respect to the straight line $x_o + \mathbb{R}\nu$, in the sense that

$$w(x) = w(\mathfrak{S}(x)) \quad \text{for all } x \in \mathcal{R}.$$

This in turn implies that

$$\nabla w(\xi) \cdot \nu(\xi) = 0 \quad \text{for all } x \in x_o x_3 .$$

Due to the \mathbf{C}^∞ regularity of the boundary of \mathcal{R} at x_o , w is of class \mathbf{C}^∞ in a neighborhood of x_o . By uniqueness, $w = u$ on $\bar{\mathcal{R}}$. Hence, u is of class \mathbf{C}^∞ also in a neighborhood of x_o restricted to Ω .

If $x_o \in (\Gamma_c \setminus \bar{\Gamma}_e)$, to prove the regularity of u at x_o we proceed as above, simply replacing the Dirichlet condition on $x_o x_1$ by a homogeneous Neumann one, applying again Lax–Milgram Lemma and concluding by symmetry and uniqueness.

★ (u.6): the characteristic equation $\det(D^2 u(\bar{x}) - \lambda I) = 0$ in the case of a 2-dimensional problem is a quadratic equation with real solutions $\lambda_1(\bar{x}), \lambda_2(\bar{x})$ satisfying

$$\lambda_1(\bar{x}) \lambda_2(\bar{x}) = \det D^2 u(\bar{x}) , \quad \lambda_1(\bar{x}) + \lambda_2(\bar{x}) = \Delta u(\bar{x}) .$$

Note that by the \mathbf{C}^2 regularity of u proved at **(u.1)**, the equation $u = \delta^2 \Delta u$ is satisfied in whole $\bar{\Omega}$. By **(u.2)**, $\lambda_1(\bar{x}) + \lambda_2(\bar{x}) = \delta^{-2} u(\bar{x}) > 0$, so that at least one of the eigenvalues has to be (strictly) positive. \square

Proof of Proposition 2.2. By (4.2) and straightforward computations it is clear that (1.1) has a solution if and only if (4.3) has a solution which is positive a.e. in Ω . Point **(u.1)** in Lemma 4.2 ensures the existence and uniqueness of a solution to (4.3). Moreover, by **(u.2)** in Lemma 4.2 this solution is strictly positive a.e. in Ω . This allows to define $\varphi = -\delta \ln u$. The remaining regularity statements and **(E.1)** follow again from Lemma 4.2 by (4.2). So as to obtain **(E.2)**, note first that $-\nabla \varphi \cdot \nu = \frac{\delta}{u} \nabla u \cdot \nu > 0$ everywhere on Γ_e by (4.2) and **(u.3)** in Lemma 4.2. Then, integrate (4.3) on Ω , use Green Theorem and again Lemma 4.2 to obtain **(E.3)**. \square

Proof of Proposition 2.4. The present proof follows from [10, Theorem 2.7]. Indeed, referring to the notation therein, we define $q(\rho) = \rho v(\rho)$ and verify the necessary assumptions.

(Ω_{3,γ}) Ω is a bounded open subset of \mathbb{R}^2 with piecewise $\mathbf{C}^{3,\gamma}$ boundary $\partial\Omega$ by **(Ω.1)** and **(Ω.2)**.

(F) This condition is immediate since in the present case we have $F \equiv 0$.

(f) In our case $f(t, x, \rho) = \rho v(\rho) w(x)$. By **(C.1)** and the assumption that w is in $(\mathbf{C}^2 \cap \mathbf{W}^{1,\infty})(\mathbb{R}; B(0, 1))$, we have that f is of class \mathbf{C}^2 and moreover

$$\partial_\rho f(t, x, \rho) = q'(\rho) w(x) , \quad \partial_{\rho\rho}^2 f(t, x, \rho) = q''(\rho) w(x) , \quad \partial_\rho \nabla \cdot f(t, x, \rho) = q'(\rho) \nabla \cdot w(x)$$

are all functions of class \mathbf{L}^∞ on $\mathbb{R}^+ \times \Omega \times [0, R_{\max}]$.

(C) This condition follows from (C.2) because in the present case $\rho_b \equiv 0$.

We then obtain

$$\begin{aligned} \|\mathcal{S}_t \rho_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} &\leq \left(\|\rho_o\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + c_2 t \right) \exp(c_1 t) && \text{by [10, Formula (2.5)]} \\ \text{TV}(\mathcal{S}_t \rho_o) &\leq (\mathcal{A}_1 + \mathcal{A}_2 t + \mathcal{A}_3 \text{TV}(\rho_o)) \exp(\mathcal{A}_4 t) && \text{by [10, Formula (6.44)]} \end{aligned}$$

where, with reference to [10, Formula (5.1)] and [10, § 6], the constants $c_1, c_2, \mathcal{A}_1, \dots, \mathcal{A}_4$ are estimated as follows:

$$\begin{aligned} c_1 &= 1 + \|q'\|_{\mathbf{L}^\infty([0, R_{\max}]; \mathbb{R})} \|\nabla \cdot w\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq 1 + \|q\|_{\mathbf{W}^{1, \infty}([0, R_{\max}]; \mathbb{R})} \|w\|_{\mathbf{W}^{1, \infty}(\Omega; \mathbb{R})} , \\ c_2 &= 0 , \\ \mathcal{A}_1 &= \mathcal{O}(1) \|Df\|_{\mathbf{L}^\infty(\Omega \times [0, R_{\max}]; \mathbb{R}^{n \times (1+n)})} \leq \mathcal{O}(1) \|q\|_{\mathbf{W}^{1, \infty}([0, R_{\max}]; \mathbb{R})} \|w\|_{\mathbf{W}^{1, \infty}(\Omega; \mathbb{R}^n)} , \\ \mathcal{A}_2 &= \mathcal{O}(1) \|Df\|_{\mathbf{W}^{1, \infty}(\Omega \times [0, R_{\max}]; \mathbb{R}^{n \times (1+n)})} \leq \mathcal{O}(1) \|q\|_{\mathbf{W}^{2, \infty}([0, R_{\max}]; \mathbb{R})} \|w\|_{\mathbf{W}^{2, \infty}(\Omega; \mathbb{R}^n)} , \\ \mathcal{A}_3 &= \mathcal{O}(1) + \|q'\|_{\mathbf{L}^\infty([0, R_{\max}]; \mathbb{R})} \|w\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^n)} \leq \mathcal{O}(1) + \|q\|_{\mathbf{W}^{1, \infty}([0, R_{\max}]; \mathbb{R})} \|w\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^n)} , \\ \mathcal{A}_4 &= \mathcal{O}(1) \left[1 + \|Df\|_{\mathbf{W}^{1, \infty}(\Omega \times [0, R_{\max}]; \mathbb{R}^{n \times (1+n)})} \right] \leq \mathcal{O}(1) \left[1 + \|q\|_{\mathbf{W}^{2, \infty}([0, R_{\max}]; \mathbb{R})} \|w\|_{\mathbf{W}^{2, \infty}(\Omega; \mathbb{R}^n)} \right] \end{aligned}$$

and the above norms of q are bounded by (C.1) and by the adopted assumption on w . \square

For technical reasons, below we fix an arbitrary open subset Ω' of \mathbb{R}^2 containing $\bar{\Omega}$ and extend the unique generalized solution $\varphi \in \mathbf{C}^3(\bar{\Omega}; \mathbb{R})$ of (1.1) given in Proposition 2.2 introducing a map $\tilde{\varphi} \in \mathbf{C}_c^3(\mathbb{R}^2; \mathbb{R})$ such that $\tilde{\varphi} \equiv \varphi$ in Ω and $\tilde{\varphi} \equiv 0$ in $\mathbb{R}^2 \setminus \Omega'$. This is possible thanks to the regularity of φ and to the following result.

Lemma 4.3 ([15, Lemma 6.37]) *Let Ω satisfy (Ω.1), (Ω.2), (Ω.3). For any open subset Ω' of \mathbb{R}^2 such that $\bar{\Omega} \subset \Omega'$, there exists a constant C such that for any $f \in \mathbf{C}^3(\Omega; \mathbb{R})$, there exists a map $\tilde{f} \in \mathbf{C}_c^3(\mathbb{R}^2; \mathbb{R})$ with*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for all } x \in \Omega \\ 0 & \text{for all } x \in \mathbb{R}^2 \setminus \Omega' \end{cases} \quad \text{and} \quad \|\tilde{f}\|_{\mathbf{C}^3(\mathbb{R}^2; \mathbb{R})} \leq C \|f\|_{\mathbf{C}^3(\bar{\Omega}; \mathbb{R})} .$$

Proof of Proposition 3.1. First, apply Lemma 4.3 and extend φ to a $\tilde{\varphi} \in \mathbf{C}^3(\mathbb{R}^2; \mathbb{R})$.

Define $\tilde{w}(x) = \mathcal{N}(-\nabla \tilde{\varphi}(x))$. By (4.1), Lemma 4.3 and Proposition 2.2, $\tilde{w} \in \mathbf{C}^{0,1}(\mathbb{R}^2; \mathbb{R}^2)$. Hence, for any fixed $\hat{x} \in \mathbb{R}^2$, the Cauchy problem

$$\dot{x} = \tilde{w}(x) , \quad x(0) = \hat{x} \quad (4.5)$$

admits a unique solution $\tilde{p}_{\hat{x}} : \mathbb{R} \rightarrow \mathbb{R}^2$. Define

$$T_{\hat{x}} = \sup \left\{ t \in \mathbb{R}^+ : \tilde{p}_{\hat{x}}([0, t]) \subset \Omega \right\} \quad \text{and} \quad p_{\hat{x}}(t) = \tilde{p}_{\hat{x}}(t) \quad \text{for } t \in [0, T_{\hat{x}}] .$$

By construction, the map $p_{\hat{x}}$ solves (3.1). By the standard theory of ordinary differential equations, (Q.1) and (Q.2) are proved.

We consider now (Q.3). Note that (4.5) is dissipative in Ω , in the sense that $\tilde{\varphi}$ is a (strict) Lyapunov function for (4.5) in Ω , i.e., $\tilde{\varphi}$ decreases along the path $t \rightarrow p_{\hat{x}}(t)$ as long as $p_{\hat{x}}(t) \in \Omega$. In fact, as long as $p_{\hat{x}}(t) \in \Omega$

$$\frac{d}{dt} \tilde{\varphi}(p_{\hat{x}}(t)) = \frac{d}{dt} \varphi(p_{\hat{x}}(t)) = - \left(\vartheta^2 + \|\nabla \varphi(p_{\hat{x}}(t))\|^2 \right)^{-1/2} \|\nabla \varphi(p_{\hat{x}}(t))\|^2 ,$$

which is strictly negative whenever \hat{x} is not a critical point. By La Salle Principle [16, Theorem 9.22, see also Lemma 9.21 and Theorem 14.17], as t goes to infinity, every bounded path $p_{\hat{x}}$ that remains in Ω is attracted towards the set of equilibria, i.e., of critical points of (4.5). More precisely, setting

$$\omega(\hat{x}) = \left\{ x \in \mathbb{R}^2 : \begin{array}{l} \text{there exists } (t_n)_{n \in \mathbb{N}} \text{ such that} \\ \lim_{n \rightarrow \infty} t_n = \infty \text{ and } \lim_{n \rightarrow \infty} p_{\hat{x}}(t_n) = x \end{array} \right\} , \quad \mathcal{E}_D = \{x \in D : \nabla \tilde{\varphi}(x) = 0\} \quad \text{for } D \subseteq \mathbb{R}^2$$

we proved that if $x \in \omega(\hat{x}) \cap \Omega$ for a $\hat{x} \in \Omega$, then $\nabla \varphi(x) = 0$.

Note that for any $\hat{x} \in \Omega$, the path $\tilde{p}_{\hat{x}}$ exiting \hat{x} does not intersect Γ_w . Indeed, by the boundary condition imposed along Γ_w in (1.1)

$$\Gamma_w = \{x \in \Gamma_w : \nabla \varphi(x) = 0\} \cup \{x \in \Gamma_w : \nabla \varphi(x) \neq 0 \text{ and } \nabla \varphi(x) \cdot \nu(x) = 0\} .$$

The former set above is clearly invariant, both positively and negatively, with respect to (4.5), hence it can not be reached by a path $t \rightarrow p_{\hat{x}}(t)$ starting in Ω . The latter consists of trajectories solving (4.5) that are entirely contained in Γ_w , since w is parallel to Γ_w . As a consequence, for any $\hat{x} \in \Omega$, either the path $t \rightarrow p_{\hat{x}}(t)$ crosses Γ_e , or it stays in Ω and approaches a point in the set $\mathcal{E}_{\bar{\Omega}}$, namely $\omega(\hat{x}) \subseteq \mathcal{E}_{\bar{\Omega}}$.

It remains to determine the behaviour of the system near the critical points in $\mathcal{E}_{\bar{\Omega}}$. We proceed by linearisation around \bar{x} , with $\nabla\varphi(\bar{x}) = 0$. Denote by $A(\bar{x})$ the first order total derivative of $\mathcal{N}(-\nabla\varphi)$ computed at $\bar{x} \in \mathcal{E}_{\bar{\Omega}}$. By direct computations,

$$A(\bar{x}) = D\mathcal{N}(-\nabla\varphi(\bar{x})) = -\vartheta^{-1} D^2\varphi(\bar{x}), \quad (4.6)$$

thanks to $\nabla\varphi(\bar{x}) = 0$. Recall the map u given by (4.2). Due to (4.3) and (4.6) we have

$$A(\bar{x}) = \frac{1}{\vartheta} \frac{\delta}{u(\bar{x})} D^2u(\bar{x}),$$

proving that $A(\bar{x})$ is symmetric and diagonalizable. By **(u.6)** in Lemma 4.2, $A(\bar{u})$ has at least one strictly positive eigenvalue, say $\lambda_2 > 0$. Consider now two cases, depending on the value attained by the other eigenvalue λ_1 :

★ $\lambda_1 \neq 0$: Then, by Hartman-Grobman Theorem, see e.g. [16, Theorem 9.35], depending on the sign of λ_1 , \bar{x} is either a source or a saddle. In both cases, it is an isolated point of $\mathcal{E}_{\bar{\Omega}}$, so that $\bar{x} \in \omega(\hat{x})$ implies $\{\bar{x}\} = \omega(\hat{x})$, by the connectedness of $\omega(\hat{x})$. This is possible only if $\lambda_1 < 0$, i.e., \bar{x} is a saddle, and \hat{x} belongs to the stable manifold consisting of two trajectories entering \bar{x} , which is a set of measure zero.

★ $\lambda_1 = 0$: Then, \bar{x} is not necessarily an isolated point of $\mathcal{E}_{\bar{\Omega}}$. We use here the result of Palmer [22] about the local central manifold, which is an invariant 1-dimensional set containing all possible critical points in a neighborhood of \bar{x} . This result can be seen as a generalization of the Hartman-Grobman Theorem, and gives the instability of the central manifold, see also [3, § 4], [7, § 9.2-9.3], [16, Theorem 10.14].

Let B be the change of coordinates matrix such that $B A(\bar{x}) B^{-1}$ is diagonal, with $A(\bar{x})$ given in (4.6). By means of the linear change of variables $y(t) = B(p_{\hat{x}}(t) - \bar{x})$, the differential equation in (4.5) can be written as

$$\dot{y}_1 = f_1(y_1, y_2), \quad \dot{y}_2 = \lambda_2 y_2 + f_2(y_1, y_2), \quad (4.7)$$

where $f \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R}^2)$ is bounded, see Lemma 4.3, and satisfies $f(0) = 0$. The dependence of B , f and λ_2 upon \bar{x} is here neglected. We obtain from [22] that there exist a Lipschitz continuous function h and a homeomorphism $H: \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that the graph of h is the local central manifold and the map $z(t) = H(t, y(t))$, with $H(t, 0) = 0$, solves

$$\dot{z}_1 = f_1(z_1, h(t, z_1)), \quad \dot{z}_2 = \lambda_2 z_2, \quad (4.8)$$

provided y solves (4.7). As a matter of fact, h can be proved to be also \mathbf{C}^2 , see [3, Proposition 4.1] or [16, Theorem 10.14].

Then, by continuity of H , there exists $r_0 > 0$ such that if $\|y(t)\| < r_0$, then $|z_2(t)| = |H_2(t, y(t))| < |z_2(0)|$. Solving the second equation in (4.8), we obtain that for $y(0)$ such that $z_2(0) = H_2(0, y(0)) \neq 0$, there exists $t_* > 0$ such that $\|y(t)\| > r_0$ for all $t > t_*$. Going back to the original x -variable, for any neighborhood \mathcal{O} of \bar{x} with $\mathcal{O} \subseteq \mathbb{R}^2$, introduce $W = \{x \in \mathcal{O}: H_2(0, B(x - \bar{x})) = 0\}$. We have obtained that if $\hat{x} \in \mathcal{O} \setminus W$, then $p_{\hat{x}}(t)$ is outside \mathcal{O} for all $t > t_*$. Thus, \bar{x} can be attractive only for the points lying on W , which is clearly a 1-dimensional manifold and has 2-dimensional Lebesgue measure equal to 0. Moreover, W as a whole is repulsive.

Therefore, $\omega(\hat{x}) \cap W$ is non-empty only if the path passing through \hat{x} lies inside W . Therefore, the 1-dimensional Lebesgue measure of $\omega(\hat{x}) \cap W$ is 0.

Finally, for almost all \hat{x} , the path $p_{\hat{x}}(\mathbb{R}^+)$ given by (4.5) is not attracted by $\mathcal{E}_{\bar{\Omega}}$, hence it has to reach the exit Γ_e , i.e., there exists a positive finite time $T_{\hat{x}}$ such that $p_{\hat{x}}(T_{\hat{x}}) \in \Gamma_e$. \square

Acknowledgment: The authors were supported by the INDAM-GNAMPA project *Leggi di conservazione nella modellizzazione di dinamiche di aggregazione*. The last author was partially supported by ICM, UW.

References

- [1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Communications on Pure and Applied Mathematics*, 12(4):623–727, 1959.

- [2] D. Amadori, P. Goatin, and M. D. Rosini. Existence results for Hughes' model for pedestrian flows. *J. Math. Anal. Appl.*, 420(1):387–406, 2014.
- [3] B. Aulbach. *Continuous and discrete dynamics near manifolds of equilibria*, volume 1058. Springer Berlin, 1984.
- [4] C. Bardos, A. Y. le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.
- [5] R. Borsche, R. M. Colombo, M. Garavello, and A. Meurer. Differential equations modeling crowd interactions. *Journal of Nonlinear Science*, pages 1–33, 2015.
- [6] L. Bruno, A. Tosin, P. Tricerri, and F. Venuti. Non-local first-order modelling of crowd dynamics: A multidimensional framework with applications. *Applied Mathematical Modelling*, 35(1):426 – 445, 2011.
- [7] S.-N. Chow and J. K. Hale. *Methods of bifurcation theory*, volume 251. New York [etc.]: Springer, 1982.
- [8] R. M. Colombo, M. Garavello, and M. Mercier. A class of nonlocal models for pedestrian traffic. *Mathematical Models and Methods in Applied Sciences*, 22(04):1150023, 2012.
- [9] R. M. Colombo and M. Lécureux-Mercier. Nonlocal crowd dynamics models for several populations. *Acta Mathematica Scientia*, 32(1):177–196, 2012.
- [10] R. M. Colombo and E. Rossi. Rigorous estimates on balance laws in bounded domains. *Acta Mathematica Scientia*, 35(4):906 – 944, 2015.
- [11] E. Cristiani, F. Priuli, and A. Tosin. Modeling rationality to control self-organization of crowds: An environmental approach. *SIAM Journal on Applied Mathematics*, 75(2):605–629, 2015. cited By 0.
- [12] M. Di Francesco, P. A. Markowich, J.-F. Pietschmann, and M.-T. Wolfram. On the Hughes' model for pedestrian flow: The one-dimensional case. *Journal of Differential Equations*, 250(3):1334–1362, 2011.
- [13] N. El-Khatib, P. Goatin, and M. D. Rosini. On entropy weak solutions of hughes' model for pedestrian motion. *Zeitschrift für angewandte Mathematik und Physik*, 64(2):223–251, 2013.
- [14] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [15] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer Science & Business Media, 2001. reprint of the 1998 edition.
- [16] J. K. Hale and H. Koçak. *Dynamics and bifurcations*, volume 3. Springer-Verlag, New York, 1991.
- [17] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research Part B: Methodological*, 36(6):507 – 535, 2002.
- [18] R. L. Hughes. The flow of human crowds. *Annual Review of Fluid Mechanics*, 35(1):169–182, 2003.
- [19] Y. Jiang, S. Zhou, and F.-B. Tian. Macroscopic pedestrian flow model with degrading spatial information. *Journal of Computational Science*, 10:36 – 44, 2015.
- [20] P. Kachroo. *Pedestrian Dynamics: Mathematical Theory and Evacuation Control*. CRC Press, 2009.
- [21] J. Oddson. On the boundary point principle for elliptic equations in the plane. *Bulletin of the American Mathematical Society*, 74(4):666–670, 1968.
- [22] K. J. Palmer. Linearization near an integral manifold. *Journal of Mathematical Analysis and Applications*, 51(1):243–255, 1975.
- [23] M. H. Protter and H. F. Weinberger. *Maximum principles in differential equations*. Springer, 1984.
- [24] A. Sard et al. The measure of the critical values of differentiable maps. *Bull. Amer. Math. Soc.*, 48(12):883–890, 1942.
- [25] M. Twarogowska, P. Goatin, and R. Duvigneau. Macroscopic modeling and simulations of room evacuation. *Applied Mathematical Modelling*, 38(24):5781 – 5795, 2014.
- [26] E. A. Volkov. Differentiability properties of solutions of boundary value problems for the Laplace and Poisson equations on a rectangle. *Trudy Matematicheskogo Instituta im. VA Steklova*, 77:89–112, 1965.
- [27] A. I. Vol'pert. Spaces BV and quasilinear equations. *Mat. Sb. (N.S.)*, 73 (115):255–302, 1967.

INDAM Unit, c/o DII, University of Brescia, Via Branze 38, 25123 Brescia, Italy

E-mail: rinaldo.colombo@unibs.it

ICM, University of Warsaw, ul. Prosta 69, P.O. Box 00-838, Warsaw, Poland

E-mail: M.Gokieli@icm.edu.pl

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, Plac Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland

E-mail: mrosini@umcs.lublin.pl